Linear Algebra

Determinant

COMPRESSED VERSION

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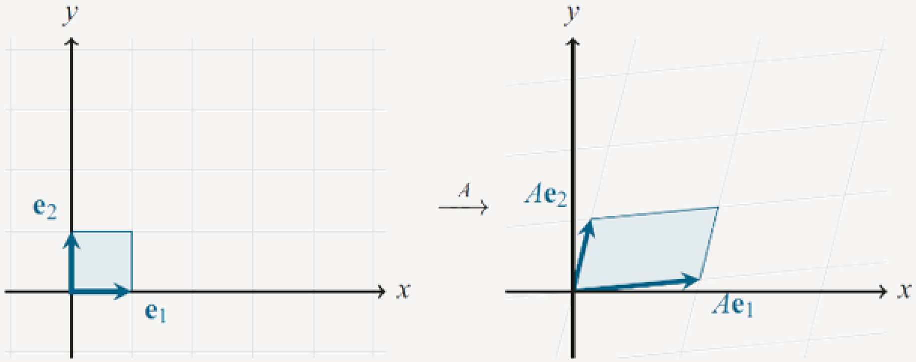
Geometric interpretation

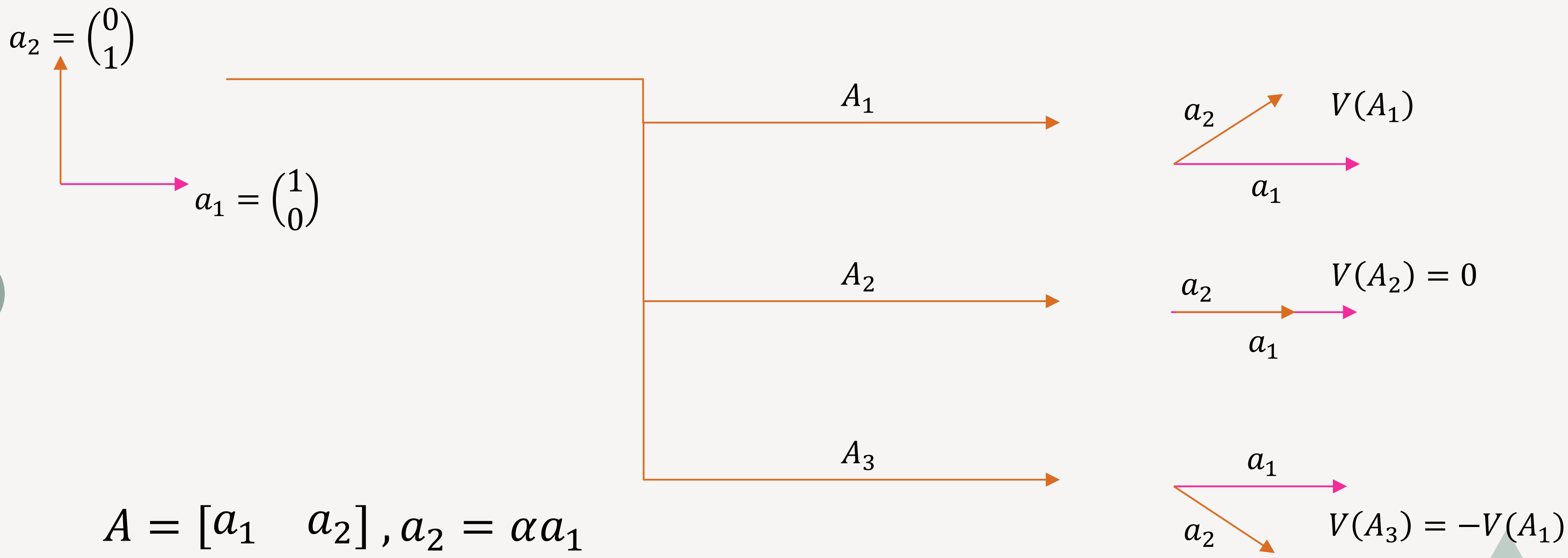
- The volume is a n-alternating multilinear map on all n-parallelepipeds such that the volume of standard unit parallelepiped is one.

$$\frac{\textit{volume of output region}}{\textit{volume of input region}}$$



A 2×2 matrix A stretches the unit square (with sides e_1 and e_2) into a parallelogram with sides Ae_1 and Ae_2 (the columns of A). The determinant of A is the area of this parallelogram.

Geometric interpretation

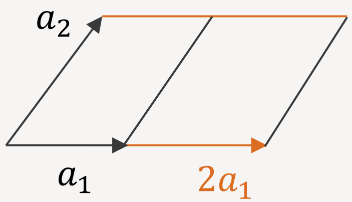


$$A = [a_1 \quad a_2]\,, a_2 = \alpha a_1$$

$$V(a_1, a_2) = -V(a_2, a_1)$$

Volume Example

- 2D Case
  - V(a_1, a_2)
  - V(2 a_1,a_2) = 2V(a_1, a_2)
  - V(-a_1, a_2) = - V(a_1, a_2)
  - V(βa_2, a_1) = - β V(a_1, a_2)



Bilinear Form over a complex vector

Definition

Suppose V and W are vector spaces over the same field $\mathbb{C}$. Then a function $\alpha: V \times W \rightarrow \mathbb{C}$ is called a **bilinear form** if it satisfies the following properties:

It is linear in its first argument:

$\alpha(\mathbf{v_1} + \mathbf{v_2}, \mathbf{w}) = \alpha(\mathbf{v_1}, \mathbf{w}) + \alpha(\mathbf{v_2}, \mathbf{w})$ and

$\alpha\,(\lambda\mathbf{v_1}, \mathbf{w}) = \lambda\alpha(\mathbf{v_1}, \mathbf{w})$ for all $\lambda \in \mathbb{C}, \mathbf{v_1}, \mathbf{v_2} \in V$, and $\mathbf{w} \in W$.

It is conjugate linear in its second argument:

$\alpha\,(\mathbf{v}, \mathbf{w_1} + \mathbf{w_2}) = \alpha(\mathbf{v}, \mathbf{w_1}) + \alpha(\mathbf{v}, \mathbf{w_2})$ and

$\alpha\,(\mathbf{v}, \lambda\mathbf{w_1}) = \bar{\lambda}\alpha(\mathbf{v}, \mathbf{w_1})$ for all $\lambda \in \mathbb{C}, \mathbf{v} \in V$, and $\mathbf{w_1}, \mathbf{w_2} \in W$.

The set of bilinear forms on $\mathbf{v}$ is denoted by $\mathbf{v}^2$.

Alternating bilinear form

Definition

A bilinear form $\alpha \in V^{(2)}$ is called alternating if

$$\alpha(v, v) = 0$$

for all $v \in V$. The set of alternating bilinear forms on V is denoted by $V_{alt}^{(2)}$.

Multilinear Forms

Definition

Suppose $\mathcal{V}_1, \mathcal{V}_2, \dots, \mathcal{V}_p$ are vector spaces over the same field $\mathbb{F}$. A function

$$f : \mathcal{V}_1 \times \mathcal{V}_2 \times \cdots \times \mathcal{V}_p \to \mathbb{F}$$

is called a multilinear form ($\mathrm{f} \in \mathrm{V}^{(\mathrm{p})}$) if, for each $1 \le j \le p$ and each $\mathrm{v}_1 \in \mathcal{V}_1, \mathrm{v}_2 \in \mathcal{V}_2, \dots, \mathrm{v}_p \in \mathcal{V}_p$, it is the case that the function $g : \mathcal{V}_j \to \mathbb{F}$ defined by

$$g(\mathrm{v}) = f\left(\mathrm{v}_1, \dots, \mathrm{v}_{j-1}, \mathrm{v}, \mathrm{v}_{j+1}, \dots, \mathrm{v}_p\right) \qquad for\ all \quad \mathrm{v} \in \mathcal{V}_j$$

is a linear form.

Multilinear Forms

Definition

Suppose m is a positive integer.

- An m−linear form α on V is called ***alternating*** if $\alpha(v_1, \dots, v_m) = 0$ whenever $v_1, \dots, v_m$ is a list of vectors in V with $v_j = v_k$ for some two distinct values of j and k in $\{1, \dots, m\}$.

- $V_{alt}^{(m)} = \{\alpha \in V^{(m)} : \alpha \text{ is an alternating } m\text{−linear form on } V\}$.

Alternating multilinear forms and linear dependence

Theorem

Suppose m is a positive integer and α is an alternating m-linear form on V. If $v_1, \dots, v_m$ is a linearly dependent list in V, then

$$\alpha(v_1, \dots, v_m) = 0$$

Proof

Swapping input vectors in an alternating multilinear form

Theorem

Suppose m is a positive integer, α is an alternating m-linear form on V, and $v_1, \dots, v_m$ is a list of vectors in V. Then swapping the vectors in any switch of $\alpha(v_1, \dots, v_m)$ changes the value of α by a factor of -1.

Okey, clearing up the last detail. Suppose we know that A(e_1, e_2, e_3, e_4, e_5) = 7. What should A(e_3, e_5, e_1, e_2, e_4) be?

A(e_3, e_5, e_1, e_2, e_4) = - A(e_3, e_4, e_1, e_2, e_5)
= A(e_3, e_2, e_1, e_4, e_5)
= - A(e_1, e_2, e_3, e_4, e_5) = -7

What if we did the switching in a different order? Would we get the same sign? It turns out that, yes, we would!

Example

- What we need to show is that there is a way to assign a sign to every permutation of $\{1, 2, 3, \ldots, k\}$ such that, switching the order of any two elements, switches the sign. For example:

$$\begin{array}{rrrr} (1,2,3) \rightsquigarrow & 1 & (1,3,2) \rightsquigarrow & -1 \\ (2,1,3) \rightsquigarrow & -1 & (2,3,1) \rightsquigarrow & 1 \\ (3,1,2) \rightsquigarrow & 1 & (3,2,1) \rightsquigarrow & -1 \end{array}$$

Here is the rule: The sign of $(\sigma(1), \sigma(2), \ldots, \sigma(k))$ is

$$(-1)^{\#\{(i,j)\ :\ i<j \text{ and } \sigma(i)>\sigma(j)\}}.$$

$$A(e_{j_1}, e_{j_2}, \ldots, e_{j_k}) = \mathtt{sign}(\sigma) A(e_{j_{\sigma(1)}}, e_{j_{\sigma(2)}}, \ldots, e_{j_{\sigma(k)}}).$$

Definition

- In **Theorem (9)** if we consider $\alpha(e_1,..,e_n) = 1$, then we can say that determinant is a multilinear alternating form (volume of the transformed vectors.)
- **Determinant is defined for square matrices.**

Definition

Suppose that n is a positive integer and A is an n-by-n square matrix. Then:

$$detA = \sum_{(j_1,\ldots,j_n)\in perm(n)} (sign(j_1,\ldots,j_n))A_{j_1,1}\ldots A_{j_n,n}$$

Definition

Note

Let $T: \mathbb{R}^2 \to \mathbb{R}^2$ be the linear transformation determined by a 2 $\times$ 2 matrix A. If S is a parallelogram in $\mathbb{R}^2$, then

$$\{area\ of\ T(S)\} = |\det A| . \{area\ of\ S\}$$

If T is determined by a 3×3 matrix A, and if S is a parallelepiped in $\mathbb{R}^3$, then

$$\{volume\ of\ T(S)\} = |\det A| . \{volume\ of\ S\}$$

Example

Example

Determinant of 2*2 matrix

Determinant of 3*3 matrix

$V(\begin{bmatrix} a & c \\ b & d \end{bmatrix}) = V(\begin{bmatrix} a \\ b \end{bmatrix}, \begin{bmatrix} c \\ d \end{bmatrix})$

$V(a\begin{bmatrix} 1 \\ 0 \end{bmatrix} + b\begin{bmatrix} 0 \\ 1 \end{bmatrix}, c\begin{bmatrix} 1 \\ 0 \end{bmatrix} + d\begin{bmatrix} 0 \\ 1 \end{bmatrix})$

$a\,V(\begin{bmatrix} 1 \\ 0 \end{bmatrix}, c\begin{bmatrix} 1 \\ 0 \end{bmatrix} + d\begin{bmatrix} 0 \\ 1 \end{bmatrix}) + bV(\begin{bmatrix} 0 \\ 1 \end{bmatrix}, c\begin{bmatrix} 1 \\ 0 \end{bmatrix} + d\begin{bmatrix} 0 \\ 1 \end{bmatrix})$

$ac\,V(\begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \end{bmatrix}) + ad\,V(\begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix}) + bc\,V(\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}) + bd\,V(\begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix}) + 0 + ad\,V(\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}) + bc\,V(\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix})$

$+ 0$

$= ad - bc\,V(\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}) = ad - bc$

$\Rightarrow V(\begin{bmatrix} a & c \\ b & d \end{bmatrix}) = ad - bc$

Famous Formula of Determinant

Determinant

Example

Let $n = \dim V$.

- If I is the identity operator on V, then $\alpha_1 = \alpha$ for all $\alpha \in V_{alt}^{(n)}$. Thus, $\det I = 1$.
- More generally, if $\lambda \in \boldsymbol{F}$, then $\alpha_{\lambda I} = \lambda^n \alpha$ for all $\alpha \in V_{alt}^{(n)}$. Thus, $\det(\lambda I) = \lambda^n$.
- Still more generally, if $T \in \mathcal{L}(V)$ and $\lambda \in \boldsymbol{F}$, then $\alpha_{\lambda T} = \lambda^n \alpha_T = \lambda^n (\det T)\alpha$ for all $\alpha \in V_{alt}^{(n)}$. Thus, $\det(\lambda T) = \lambda^n \det T$.

Gaussian Elimination & Determinant

Note

- Row operations
- Let A be a square matrix.
- If a multiple of one row of A is added to another row to produce a matrix B, then $\det(B) = \det(A)$
- If two rows of A are interchanged to produce B, then $\det(B) = -\det(A)$
- If one row of A is multiplied by k to produce B, then $\det(B) = k.\det(A)$

Properties

- (1) If one row or column is zero, then determinant is zero

$$\begin{vmatrix} 0 & 0 & 0 \\ a & b & c \\ d & e & f \end{vmatrix} = 0$$

  - Determinant of zero matrix is…

$$\det(A) = \sum_{j=1}^{n} (-1)^{1+j} a_{1j} \det(A_{1j})$$

$$\det(A) = \sum_{\sigma \in S_n} \operatorname{sgn}(\sigma) \prod_{i=1}^{n} a_{i\sigma(i)}$$

Properties

- (2) If two rows or columns of matrix are same, then determinant is zero.

$$A = \begin{bmatrix} 1 & -2 & 3 \\ 1 & -2 & 3 \\ 5 & 3 & -1 \end{bmatrix}$$

Properties

- (3) If two rows or columns of matrix are interchanged, the sign of determinant is changes!
- $(4)\ \det(I) = 1$

Properties

- (5) Row and Column Operations
  - If a multiple of one row/column of A is added to another row/column to produce a matrix B, then $\det(A) = \det(B)$.

Proof?

Example

$$\begin{vmatrix} 1 & -1 & 2 \\ 0 & 2 & -3 \\ 0 & 0 & -2 \end{vmatrix} = \begin{vmatrix} 1 & -1 & 2 \\ 1 & 1 & -1 \\ 1 & -1 & 0 \end{vmatrix} = \begin{vmatrix} 1 & -1 & 6 \\ 1 & 1 & 3 \\ 1 & -1 & 4 \end{vmatrix}$$

Properties

- (6) If A is a triangular matrix, then $\det(A)$ is the product of the entries on the main diagonal of A.

$$\begin{vmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{vmatrix} = abc \qquad \begin{vmatrix} a & 0 & 0 \\ d & b & 0 \\ e & f & c \end{vmatrix} = abc$$

  - Determinant of identity matrix is…

- U is unitary, so that |det(U)|=I

Properties

- (7) If a column or row is multiply to k then determinant is multiply to k.

$$\begin{vmatrix} ka_{11} & \cdots & ka_{1n} \\ \vdots & \ddots & \vdots \\ a_{n1} & \cdots & a_{nn} \end{vmatrix} = k \begin{vmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{n1} & \cdots & a_{nn} \end{vmatrix}$$

- $|kA_{n\times n}| = k^n |A_{n\times n}|$

Properties

- (8) If a row/column is multiple of another row/column then determinant is .....

Properties

- (9) If columns/rows of matrix are linear dependent then its determinant is zero
- (10) If columns/rows of matrix are linear dependent if and only if its determinant is zero.
- (11) Transposing a matrix does not change the determinant.

Reference

- Chapter 3: Linear Algebra and Its Applications, David C. Lay.
- Chapter 9: Part B and C: Linear Algebra Done Right, Sheldon Axler.